A correct proof of the McMorris-Powers' theorem on the consensus of phylogenies

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Abstract

McMorris and Powers proved an Arrow-type theorem on phylogenies given as collections of quartets. There is an error in one of the main lemmas used to prove this theorem. However, this lemma (and thereby the theorem) is still true, and we provide a corrected proof.

1 Theorem of McMorris and Powers

In 1951, K. Arrow [1] proved his impossibility theorem for the aggregation of weak orders. In the past thirty years, there has been increasing interest in studying limitations and possibilities in aggregation or consensus models in many areas of science other than the social choice theory. One such study that is of interest to phylogeneticists is an impossibility theorem about trees by McMorris and Powers, (see [4]). The purpose of this note is to point out an error in the original proof, and propose a workaround.

The notation and definitions summarised below closely follow Day and McMorris [3]. Let S be an n-element set, where $n \geq 5$. A *phylogeny* on S is an unrooted tree with no vertex of degree 2, and exactly n vertices of degree 1 (*leaves*), each labelled by a distinct element of S. Let $\mathcal{P}$ denote the set of all phylogenies on S. Let $w, x, y, z \in S$. We say that the configuration (*quartet*) $wx|yz$ is in phylogeny T if the path from w to x has no vertex in common with the path from y to z. If the $w-x$ and $y-z$ paths have exactly one vertex in common then we say that the quartet $wxyz$ is in T. Any

four elements w, x, y, z occur in T in one of the four configurations $wx|yz$, $wy|zx$, $wz|xy$ (called the *resolved quartets*) and $wxyz$ (called an *unresolved quartet*). Since a tree is uniquely determined by its collection of quartets, (see [2]), we overload the notation T to sometimes denote the set of quartets $q(T)$ of T. For tree $T \in \mathcal{P}$ and $X \subseteq S$, $T|_X$ (the *restriction* of T to X) denotes the set of quartets of T made up entirely with elements from X. Similarly the restriction of a profile $P = (T_1, T_2, \ldots, T_k) \in \mathcal{P}^k$ to X is simply $P|_X = (T_1|_X, T_2|_X, \ldots, T_k|_X)$. For $w, x, y, z \in S$, the set of individuals K, $|K| = k$, and a consensus rule $C : \mathcal{P}^k \to \mathcal{P}$, the following shortcut notation is used: $wx|yz \in q(T_i)$ is denoted by wxT_iyz, $wxyz \in q(T_i)$ is denoted by $wxyzT_i$, $(\forall i \in I \subseteq K)(wxT_iyz)$ is denoted by wxT_Iyz, and $(\forall i \in I \subseteq K)(wxyzT_i)$ is denoted by $wxyzT_I$.

Definition 1. Let $C : \mathcal{P}^k \to \mathcal{P}$ be a consensus function. The notions of Dictatorship (Dct), Independence (Ind) and Pareto Optimality (PO) are defined as:
Dct :$(\exists j \in K)(\forall w, x, y, z \in S)(\forall P \in \mathcal{P}^k)(wxT_jyz \Rightarrow wxC(P)yz)$
Ind :$(\forall X \subseteq S)(\forall P, P' \in \mathcal{P}^k)(P|_X = P'|_X \Rightarrow C(P)|_X = C(P')|_X)$
PO :$(\forall w, x, y, z \in S)(\forall P \in \mathcal{P}^k)(wxT_Kyz \Rightarrow wxC(P)yz)$

In [4], McMorris and Powers proved the following Arrow-type Theorem.

Theorem 1. *Let C be a consensus rule on $\mathcal{P}$. C satisfies Dct iff it satisfies Ind and PO.*

Proof of this theorem follows Sen's strategy, (see [5] or [3]), of defining an appropriate notion of *decisiveness* for a group of individuals, (which, informally speaking, says, if a group of individuals has a certain preference then the consensus profile also imposes the same preference), followed by an *invariance* lemma for decisiveness (which, informally speaking, says, if a group of individuals is decisive about one quartet then the group is decisive about all quartets). One then refines the decisive set to prove that there exists a singleton decisive set, which is a dictatorial situation.

In the following, definitions of decisiveness are followed by the proof of the invariance lemma as presented by Day and McMorris in [3]. The proof in [4] is based on the same argument. An error in their proof is then discussed. A new proof is presented in the next section.

Definition 2. Let $C : \mathcal{P}^k \to \mathcal{P}$ be a consensus rule, $I \subseteq K$, and $wx|yz$ a quartet. I is called almost decisive for $wx|yz$, denoted by $\hat{U}^I_{wx|yz}$, if $(\forall P \in \mathcal{P}^k)(wxT_Iyz \wedge wxyzT_{K\setminus I} \Rightarrow wxC(P)yz)$. I is called almost decisive if it is almost decisive for all resolved quartets. The family of almost decisive subsets of K is denoted by $\hat{U}_C$. I is called decisive for $wx|yz$, (denoted by $U^I_{wx|yz}$) if $(\forall P \in \mathcal{P}^k)(wxT_Iyz \Rightarrow wxC(P)yz)$. I is called decisive if it is decisive for all resolved quartets. The family of decisive subsets of K is denoted by U_C.

Lemma 2. (lemma 3.34 in [3]) *Let $C : \mathcal{P}^k \to \mathcal{P}$ be a consensus rule that satisfies Ind and PO, and $I \subseteq K$. Then*

$$(\exists a, b, c, d \in S)(\hat{U}^I_{ab|cd}) \Longleftrightarrow I \in \hat{U}_C \tag{1}$$

$$(\exists a, b, c, d \in S)(U^I_{ab|cd}) \Longleftrightarrow I \in U_C \tag{2}$$

Proof Proof presented here is based on [3]. Proof of (1) in [3] is correct, but is presented here for a later reference.
Proof of (1) Let $(\exists a, b, c, d \in S)(\hat{U}^I_{ab|cd})$. Since $|S| \geq 5$, let $v \in S$ be such that $v \notin X = \{a, b, c, d\}$. We will show that $\hat{U}^I_{bv|cd}$. Construct $P \in \mathcal{P}^k$ such that
$\{ab|cd, ab|cv, ab|dv, av|cd, bv|cd\} \subseteq T_I$, and $\{abcd, av|bc, av|bd, av|cd, bcdv\} \subseteq T_{K\setminus I}$. P is otherwise unconstrained. Since $\hat{U}^I_{ab|cd}$, $ab|cd \in C(P)$. By PO, $av|cd \in C(P)$. Therefore, $bv|cd \in C(P)$. Therefore, by Ind, $\hat{U}^I_{bv|cd}$. By trivial variants of this argument, $\hat{U}^I_{wx|yz}$ is obtained for each $wx|yz$ other than $ab|cd$, thus proving $I \in \hat{U}_C$. The converse is trivial.□
Proof of (2) (original proof) Let $(\exists a, b, c, d \in S)(U^I_{ab|cd})$. Then I is also almost decisive for $ab|cd$, so by (1) it is almost decisive for all resolved quartets. We would like to prove for any $P \in \mathcal{P}^k$ and $X = \{w, x, y, z\} \subseteq S$, that $wxT_Iyz \Rightarrow wxC(P)yz$. Since $|S| \geq 5$, we can select $v \notin X$, and construct a profile P' such that $\{wx|yz, wx|vy, wx|vz, vwyz, vxyz\} \subseteq T'_I$, and $\{vwxy, vwxz\} \subseteq T'_{K\setminus I}$, and $P|_X = P'|_X$. P' is otherwise unconstrained. From (1), we have $\{wx|vy, wx|vz\} \subseteq C(P')$. Therefore, $wx|yz \in C(P')$. But $P|_X = P'|_X$, so $wx|yz \in C(P)$ as required. The converse is trivial. □

There is an error in the nontrivial direction of the proof of (2) above. Profile P' is chosen such that $\{vwxy, vwxz\} \subseteq T'_{K\setminus I}$ and $P|_X = P'|_X$. This implies $\{wxyz, wx|yz\} \cap T_{K\setminus I} \neq \emptyset$. This means, if P is such that $wy|xz \in T_{K\setminus I}$

or $wz|xy \in T_{K\backslash I}$ then no choice of P' such that $P|_X = P'|_X$, can meet the requirement $\{vwxy, vwxz\} \subseteq T'_{K\backslash I}$ of the construction. Although the result of the lemma is correct, a complete proof requires more complex arguments than the ones provided by McMorris and Powers in their original proof. In the next section, complete arguments will be provided.

2 Invariance Lemmas

The proof presented below requires four different levels of decisiveness, the first one being equivalent to almost decisiveness defined above, and the last one being the decisiveness defined above. Most proofs below follow the line of argument that can be summarised thus: we have a profile P containing a certain configuration on $X \subseteq S$. We want to prove that the configuration also occurs in the consensus profile $C(P)$. We construct a profile P' that agrees with P when restricted to X. Moreover, P' allows us to resolve the configuration in $C(P')$ using weaker notions of decisiveness and their invariance lemmas. Then by Ind, we claim that $C(P)$ contains the configuration.

Definition 3. (type-A decisiveness) Let $C : \mathcal{P}^k \to \mathcal{P}$ be a consensus rule, $I \subseteq K$, and $X = \{a, b, c, d\} \subset S$. I is called decisive-A for $ab|cd$, denoted by $A(I : C, ab|cd)$, if

$$(\forall P \in \mathcal{P}^k)(i \in I \Rightarrow ab|cd \in T_i)(i \in K\backslash I \Rightarrow abcd \in T_i)$$
$$\Rightarrow (ab|cd \in C(P))$$

I is called decisive-A, denoted by $A(I : C)$, if it is decisive-A for all resolved quartets.

Lemma 3. *Let C be a consensus rule satisfying Ind and PO, and $I \subseteq K$.*

$$(\exists a, b, c, d \in S)(A(I : C, ab|cd)) \Rightarrow A(I : C)$$

Proof The notion of decisive-A sets is equivalent to the almost decisiveness in definition 2. Also, this lemma is equivalent to the *only if* part of (1) of Lemma 2. So we skip the proof.

Definition 4. (type-B decisiveness) Let $C : \mathcal{P}^k \to \mathcal{P}$ be a consensus rule, $I \subseteq K$, and $X = \{a, b, c, d\} \subset S$. I is called decisive-B for $ab|cd$ if

$$(\forall P \in \mathcal{P}^k)(i \in I \Rightarrow ab|cd \in T_i)(i \in K\backslash I \Rightarrow \{abcd, ab|cd\} \cap T_i \neq \emptyset)$$
$$\Rightarrow (ab|cd \in C(P))$$

I is called decisive-B, denoted by $B(I:C)$, if it is decisive-B for all resolved quartets.

Lemma 4. *Let C be a consensus rule satisfying Ind and PO, and $I \subseteq K$. Then $A(I:C) \Rightarrow B(I:C)$.*

Proof Let I be a decisive-A set. Let $w, x, y, z \in S$, and $P = (T_1, T_2, \ldots, T_k)$ be any profile satisfying

1. $(i \in I \Rightarrow wx|yz \in T_i)$,
2. $(i \in K \backslash I \Rightarrow \{wxyz, wx|yz\} \cap T_i \neq \emptyset)$.

Since $|S| \geq 5$, we can select $v \notin X$, and construct a profile P' such that

1. $\{wx|yz, wx|vy, vwyz, vxyz\} \subseteq T'_I$,
2. $\{vwxy, vwxz, wxyz\} \subseteq T'_i$ whenever $wxyz \in T_i$ and $i \in K \backslash I$,
3. $\{vwxy, vwxz, wx|yz\} \subseteq T'_i$ whenever $wx|yz \in T_i$ and $i \in K \backslash I$.

P' is otherwise unconstrained. This satisfies $P|_X = P'|_X$. If $\{wx|yz, wx|vy\} \subseteq T'_I$ then we have $wx|vz \in T'_I$. Since I is decisive-A, $\{wx|vy, wx|vz\} \subseteq C(P')$. Therefore, $wx|yz \in C(P')$. By $P|_X = P'|_X$, $wx|yz \in C(P)$. Since w, x, y, z are arbitrary, I is decisive-B. $\square$

This is in fact what was proved in the original proof of part (2) of Lemma 2. Note that this is weaker than the full decisiveness that we desire.

Definition 5. (type-C decisiveness) Let $C : \mathcal{P}^k \rightarrow \mathcal{P}$ be a consensus rule, $I \subseteq K$, and $X = \{a, b, c, d\} \subset S$. I is called decisive-C for $ab|cd$ if

$$(\forall P \in \mathcal{P}^k)(i \in I \Rightarrow ab|cd \in T_i)(i \in K \backslash I \Rightarrow \{abcd, ab|cd, ac|bd\} \cap T_i \neq \emptyset) \Rightarrow (ab|cd \in C(P))$$

I is called decisive-C, denoted by $C(I:C)$, if it is decisive-C for all resolved quartets.

Lemma 5. *Let C be a consensus rule satisfying Ind and PO, and $I \subseteq K$. Then $A(I:C) \Rightarrow C(I:C)$.*

Proof Let $X = \{w, x, y, z\} \subseteq S$. Let $P = (T_1, T_2, \ldots, T_k)$ be a profile such that $wx|yz \in T_i \forall i \in I$, and $\{wxyz, wx|yz, wy|xz\} \cap T_i \neq \emptyset$ whenever $i \in K \backslash I$. We would like to prove that $wx|yz \in C(P)$. Construct a profile $P' = (T'_1, T'_2, \ldots, T'_k)$ such that

1. $\{wx|yz, wx|vy, xy|vz, wy|vz\} \subseteq T'_i$ whenever $i \in I$.

2. $\{wyvx, wyvz, wv|xz, vy|xz\} \subseteq T'_i$ whenever $wy|xz \in T_i$ and $i \in K \backslash I$.

3. $\{wx|yz, wx|vy, xy|vz, wy|vz\} \subseteq T'_i$ whenever $wx|yz \in T_i$ and $i \in K \backslash I$.

4. $\{wxyz, wxyv, wyvz, wxvz, xyzv\} \subseteq T'_i$ whenever $wxyz \in T_i$ and $i \in K \backslash I$.

P' is otherwise unconstrained. Clearly, $P|_X = P'|_X$. When $\{wx|yz, wx|vy\} \subseteq T'_i$, we have $wx|vz \in T'_i$. Similarly, if $\{wv|xz, vy|xz\} \subseteq T'_i$ then $wy|xz \in T'_i$, and if $\{xy|vz, wy|vz\} \subseteq T'_i$ then $wx|vz \in T'_i$. By Lemma 4, we have $A(I : C) \Rightarrow B(I : C)$. Applying Lemma 4 to w, y, v, z, we have $wy|vz \in C(P')$, and applying Lemma 4 to w, x, v, y, we have $wx|vy \in C(P')$. Therefore, $wx|yz \in C(P')$, and $wx|yz \in C(P)$ by Ind. Since w, x, y, z are arbitrary, I is decisive-C. $\square$

Definition 6. (type-D decisiveness) Let $C : \mathcal{P}^k \to \mathcal{P}$ be a consensus rule, $I \subseteq K$, and $X = \{a, b, c, d\} \subset S$. I is called decisive-D (or simply decisive) for $ab|cd$ if

$$(\forall P \in \mathcal{P}^k)(i \in I \Rightarrow ab|cd \in T_i) \Rightarrow (ab|cd \in C(P))$$

I is called decisive-D, or simply decisive, denoted by $D(I : C)$ if it is decisive for all resolved quartets.

Lemma 6. *Let C be a consensus rule satisfying Ind and PO, and $I \subseteq K$. Then $A(I : C) \Rightarrow D(I : C)$.*

Proof Let $A(I : C)$, so by previous lemmas, we have $B(I : C)$ and $C(I : C)$. Let $X = \{w, x, y, z\} \subseteq S$. Let $P = (T_1, T_2, \dots, T_k)$ be a profile such that $wx|yz \in T_i \forall i \in I$. P is unconstrained otherwise. Construct a profile $P' = (T'_1, T'_2, \dots, T'_k)$ such that

1. $\{wx|yz, wx|vy, wyvz, xyvz\} \subseteq T'_i$ whenever $i \in I$.

2. $\{wyvx, wyvz, wv|xz, vy|xz\} \subseteq T'_i$ whenever $wy|xz \in T_i$ and $i \in K \backslash I$.

3. $\{wzvx, wzvy, vz|xy, wv|xy\} \subseteq T'_i$ whenever $wz|xy \in T_i$ and $i \in K \backslash I$.

4. $\{wxvy, wxvz, wv|yz, xv|yz\} \subseteq T'_i$ whenever $wx|yz \in T_i$ and $i \in K \backslash I$.

5. $\{wxvy, wxvz, wxyz, wvyz, xvyz\} \subseteq T_i'$ whenever $wxyz \in T_i$ and $i \in K\backslash I$.

P' is otherwise unconstrained. Clearly, $P|_X = P'|_X$. When $\{wx|yz, wx|vy\} \subseteq T_i'$, we have $wx|vz \in T_i'$. Similarly, when $\{wv|xz, vy|xz\} \subseteq T_i'$, we have $wy|xz \in T_i'$, and if $\{vz|xy, wv|xy\} \subseteq T_i'$ then $wz|xy \in T_i'$, and if $\{wv|yz, xv|yz\} \subseteq T_i'$ then $wx|yz \in T_i'$. By Lemma 5, we have $\{wx|vy, wx|vz\} \subseteq C(P')$, which implies $wx|yz \in C(P')$ and, by Ind, $wx|yz \in C(P)$. $\square$

This lemma implies part (2) of Lemma 2.

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